

PARTITIONS WITH THE SAME HOOK MULTISSET

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ABSTRACT. It is well-known that two conjugate partitions have the same hook multiset. But two different partitions with the same hook multiset may not be conjugate to each other. In 1977, Herman and Chung proposed the following question: What are the necessary and sufficient conditions for partitions to be determined by their hook multisets up to conjugation? In this paper, we will answer this question by giving a criterion to determine whether two different partitions with the same hook multiset are conjugate to each other.

1. INTRODUCTION

Hook lengths of partitions are very useful in the study of number theory, combinatorics and representation theory. A *partition* is a finite sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$. A partition λ could be visualized by its *Young diagram*, which is a finite collection of boxes arranged in left-justified such that there are exactly λ_i boxes in the i -th row. For every box in the Young diagram of λ , we can associate its *hook length*, which is the number of boxes in the same row to the right of it, in the same column below it, or the box itself. We denote the hook length of (i, j) -box by $h(i, j)$. The *hook multiset* $H(\lambda)$ of the partition λ is defined to be the multiset of hook lengths of λ . The *conjugation* of the partition λ is defined by $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_r)$ where $r = \lambda_1$ and $\lambda'_i = \#\{1 \leq j \leq m : \lambda_j \geq i\}$ for $1 \leq i \leq r$. It is easy to see that, we can obtain the Young diagram of the conjugation of a partition by reflecting its Young diagram along its main diagonal. For example, Figure 1 shows the Young diagrams and hook lengths of partitions $(4, 2, 2)$ and $(3, 3, 1, 1)$. It is easy to see that these two partitions are conjugate to each other.

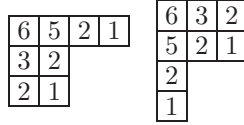


FIGURE 1. The Young diagrams and hook lengths of partitions $(4, 2, 2)$ and $(3, 3, 1, 1)$.

The β -set $\beta(\lambda)$ of the partition λ is the set of hook lengths of boxes in the first column of the corresponding Young diagram, which means that

$$\beta(\lambda) = \{\lambda_i + m - i : 1 \leq i \leq m\}.$$

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It is easy to see that a partition is uniquely determined by its β -set. If A is a finite set of some positive integers, we define λ_A to be the partition whose β -set is A .

As we know, two conjugate partitions have the same hook multiset. But in general, a partition is not uniquely determined by its hook multiset up to conjugation. For instance, Herman and Chung [3] showed that for any $n \in \mathbf{N}$, the following two non-conjugate partitions have the same hook multiset:

$$\lambda_n = (n+6, n+3, n+3, 2) \text{ and } \mu_n = (n+5, n+5, n+2, 1, 1).$$

Herman and Chung proposed the following question:

Question 1.1. ([3].) *What are the necessary and sufficient conditions for partitions to be determined by their hook multisets up to conjugation?*

Equivalently, given two different finite sets A and B of some positive integers with $H(\lambda_A) = H(\lambda_B)$, what are the necessary and sufficient conditions for partitions λ_A and λ_B to be conjugate to each other?

We will give an answer to this question:

Theorem 1.2. *Let A and B be two different subsets of $\{1, 2, \dots, n\}$ with $n \in A \cap B$ and $H(\lambda_A) = H(\lambda_B)$. Then partitions λ_A and λ_B are conjugate to each other if and only if $A \setminus B$ and $B \setminus A$ are n -symmetric sets, where a set S is called n -symmetric if $S = \{n - x : x \in S\}$.*

Example. Let $A = \{2, 3, 6\}$ and $B = \{1, 2, 5, 6\}$. In this case $\lambda_A = (4, 2, 2)$ and $\lambda_B = (3, 3, 1, 1)$. By Figure 1 we know $H(\lambda_A) = H(\lambda_B)$. Since $A \setminus B = \{3\}$ and $B \setminus A = \{1, 5\}$ are 6-symmetric sets, by Theorem 1.2 we know $\lambda_A = (4, 2, 2)$ and $\lambda_B = (3, 3, 1, 1)$ must be conjugate to each other, which is indeed true.

2. MAIN RESULTS

Suppose that n is a positive integer and A is a finite set of some positive integers. Let $I_n = \{0, 1, \dots, n\}$ and $n - A = \{n - x : x \in A\}$. The following results are well-known and easy to prove:

Lemma 2.1. ([1].) *Suppose that A is a finite set of some positive integers whose largest element is n . Let A' be the complement of A in I_n . Then the β -set of the conjugation of λ_A is $n - A'$. The hook multiset of λ_A is*

$$H(\lambda_A) = \{a - a' : a \in A, a' \in A', a > a'\}.$$

Now we can give the proof of Theorem 1.2.

Proof of Theorem 1.2. \Rightarrow : Suppose that partitions λ_A and λ_B are conjugate to each other. Let A' be the complement of A in I_n . By Lemma 2.1, we know $B = n - A'$, thus $(n - A) \cup B = I_n$ and $(n - A) \cap B = \emptyset$. Let $a \in A \setminus B$. By $a \in A$ we know $n - a \in n - A$, which means that $n - a \notin B$. By $a \notin B$ we know $a \in n - A$, which means that $n - a \in A$. Then we have $n - a \in A \setminus B$. Thus we know $A \setminus B$ is an n -symmetric set. Similarly, $B \setminus A$ is also an n -symmetric set.

\Leftarrow : Suppose that $A \setminus B$ and $B \setminus A$ are n -symmetric sets. Let A' and B' be the complements of A and B in I_n . We assume that λ_A and λ_B are not conjugate to each other. Next we will deduce a contradiction to show that this assumption couldn't be true.

Step 1: In this step we will prove that $(n - A') \setminus B$ and $B \setminus (n - A')$ are n -symmetric sets:

First let $a \in (n - A') \setminus B$.

Then we know $a \in (n - A') = n - I_n \setminus A = I_n \setminus (n - A)$. Thus we know

$$a \notin n - A = (n - (A \cap B)) \cup (n - (A \setminus B)) = (n - (A \cap B)) \cup (A \setminus B)$$

since $A \setminus B$ is an n -symmetric set, which means that $a \notin A \setminus B$. But we already know $a \notin B$. This implies that $a \notin A$, which means that $a \in A'$ and thus $n - a \in n - A'$.

On the other hand, by $a \notin n - A$ we know $a \notin n - A \cap B$. By $a \notin B$ we know $a \notin B \setminus A = n - B \setminus A$ since $B \setminus A$ is an n -symmetric set. Put these two results together, we have

$$a \notin (n - A \cap B) \cup (n - B \setminus A) = n - ((A \cap B) \cup (n - B \setminus A)) = n - B,$$

which means that $n - a \notin B$.

Now we have $n - a \in (n - A') \setminus B$ and thus $(n - A') \setminus B$ is an n -symmetric set.

Next let $b \in B \setminus (n - A')$.

Then we know $b \in B \cap (n - A)$. Thus $n - b \in A$, which means that $n - b \notin B \setminus A = n - B \setminus A$ since $B \setminus A$ is an n -symmetric set. Then we have $b \notin B \setminus A$. But we already know $b \in B$. Thus $b \in A$, which means that $n - b \notin n - A'$.

On the other hand, $b \in B$ implies that $b \notin A \setminus B = n - A \setminus B$ since $A \setminus B$ is an n -symmetric set. Then we have $n - b \notin A \setminus B$. But we already know $n - b \in A$, thus we know $n - b \in B$.

Now we have $n - b \in B \setminus (n - A')$ and thus $B \setminus (n - A')$ is an n -symmetric set.

Step 2: Suppose that $k \in \{1, 2, \dots, n\}$. Let M and N be multisets. We define M_k to be the multiplicity of k in M . We also define $M + N$ to be the multiset $\{x + y : x \in M, y \in N\}$ and $M - N$ to be the multiset $\{x - y : x \in M, y \in N\}$. By Lemma 2.1, the multiplicity of k in $H(\lambda_A)$ and $H(\lambda_B)$ are $H(\lambda_A)_k = (A - A')_k$ and $H(\lambda_B)_k = (B - B')_k$. We know $H(\lambda_A)_k = H(\lambda_B)_k$ since $H(\lambda_A) = H(\lambda_B)$. Let $C = A \setminus B$, $D = B \setminus A$ and $E = A \cap B$. We know $C = n - C$, $D = n - D$ since $A \setminus B$ and $B \setminus A$ are n -symmetric sets.

We mention that, all the following set operations in Step 2 are on multisets. Then we have

$$\begin{aligned} H(\lambda_A)_k - H(\lambda_B)_k &= (A - A')_k - (B - B')_k \\ &= (C \cup E - A')_k - (D \cup E - B')_k \\ &= (C - A')_k + (E - A')_k - (D - B')_k - (E - B')_k. \end{aligned}$$

Since $C = n - C$, $D = n - D$, we have

$$\begin{aligned} (E - A')_k - (E - B')_k &= (E - I_n \setminus (C \cup E))_k - (E - I_n \setminus (D \cup E))_k \\ &= -(E - C)_k + (E - D)_k \\ &= -(E + n - C)_{n+k} + (E + n - D)_{n+k} \\ &= -(E + C)_{n+k} + (E + D)_{n+k}. \end{aligned}$$

Then we know

$$\begin{aligned}
& H(\lambda_A)_k - H(\lambda_B)_k \\
&= (C - A')_k - (E + C)_{n+k} - (D - B')_k + (E + D)_{n+k} \\
&= (C - A')_k - (E + C)_{n+k} - (D + C)_{n+k} \\
&+ (D + n - C)_{n+k} - (D - I_n \setminus B)_k + (E + D)_{n+k} \\
&= (C + n - A')_{n+k} - (B + C)_{n+k} \\
&+ (D + n - C)_{n+k} - (D + n - I_n \setminus B)_{n+k} + (E + D)_{n+k} \\
&= (C + n - A')_{n+k} - (B + C)_{n+k} \\
&+ (D + n - C)_{n+k} - (D + I_n \setminus (n - B))_{n+k} + (E + D)_{n+k} \\
&= (C + n - A')_{n+k} - (B + C)_{n+k} + (D + n - C)_{n+k} \\
&- (D + I_n \setminus (n - D))_{n+k} + (D + (n - E))_{n+k} + (E + D)_{n+k} \\
&= (C + n - A')_{n+k} - (B + C)_{n+k} \\
&+ (D + n - C \cup E)_{n+k} - (D + I_n \setminus D)_{n+k} + (E + D)_{n+k} \\
&= (C + n - A')_{n+k} - (B + C)_{n+k} \\
&+ (D + n - A)_{n+k} - (D + I_n)_{n+k} + (D + B)_{n+k} \\
&= (C + n - A')_{n+k} - (B + C)_{n+k} \\
&- (D + I_n \setminus (n - A))_{n+k} + (D + B)_{n+k} \\
&= (C + n - A')_{n+k} - (B + C)_{n+k} \\
&- (D + n - A')_{n+k} + (D + B)_{n+k}.
\end{aligned}$$

By Lemma 2.1 we know $C \cup D$ and $((n - A') \setminus B) \cup (B \setminus (n - A'))$ are not empty sets since we already assume that λ_A and λ_B are not conjugate to each other.

Let x be the greatest element in $C \cup D$ and y be the greatest element in $((n - A') \setminus B) \cup (B \setminus (n - A'))$. We have $x \geq \frac{n}{2}$ since C and D are n -symmetric sets. By Step 1, $(n - A') \setminus B$ and $B \setminus (n - A')$ are also n -symmetric sets, then we have $y \geq \frac{n}{2}$. It is easy to see that, $x = \frac{n}{2}$ and $y = \frac{n}{2}$ couldn't be true simultaneously. Let $z = x + y - n$. Thus we have $1 \leq z \leq n$.

Then one of the following cases must be true:

- (1) $x \in C, y \in (n - A') \setminus B$;
- (2) $x \in C, y \in B \setminus (n - A')$;
- (3) $x \in D, y \in (n - A') \setminus B$;
- (4) $x \in D, y \in B \setminus (n - A')$.

For case (1), if $x \in C, y \in (n - A') \setminus B$, then it is easy to see that

$$(C + n - A')_{n+z} \geq 1$$

and

$$(B + C)_{n+z} = (D + n - A')_{n+z} = (D + B)_{n+z} = 0$$

since $x + y = n + z$, x is the greatest element in $C \cup D$ and y is the greatest element in $((n - A') \setminus B) \cup (B \setminus (n - A'))$. Then we have

$$\begin{aligned}
& H(\lambda_A)_z - H(\lambda_B)_z \\
&= (C + n - A')_{n+z} - (B + C)_{n+z} - (D + n - A')_{n+z} + (D + B)_{n+z} \geq 1,
\end{aligned}$$

which means that $H(\lambda_A) \neq H(\lambda_B)$, a contradiction!

For case (2), (3) or (4), similarly we can also deduce that $H(\lambda_A) \neq H(\lambda_B)$, which is a contradiction.

By this contradiction we know λ_A and λ_B must be conjugate to each other. We finish the proof. \square

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